Multi-objective linear programming problem under imprecise environment

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Abstract:

The problem of solving multi-objective linear-programming problems, by assuming that the decision maker has imprecise goals for objective functions, is addressed. Several methods have been proposed in literature in order to obtain efficient solutions and then Pareto optimal solutions to multi-objective programming problems under imprecise environment. Recently Jimenez and Bilbao showed that a fuzzy efficient solution may not be Pareto optimal solution in the case that one of the fuzzy goals is fully achieved. Next Wu, Liu and Lur redefined the membership function of fuzzy set theory and proposed a new two phase method. In this paper we show that this redefined membership function has not been used during computation according to definition. An example is shows that if the redefined membership functions are used properly; it may fail to generate a solution. Therefore a new function that is strictly monotonic over entire set of real numbers as well as preserve an amazing characteristic of membership function is defined and used to find Pareto optimal solution of multi objective programming problem. Classical membership function is a special case of this newly defined function. Numerical examples illustrate our procedure.

Keywords: Multi-objective programming; Decision analysis; Fuzzy mathematical programming; efficient solution; Pareto-optimal solution; Two-phase method; T-characteristic function.

1. Introduction

Let us consider the following multi-objective linear programming problem (MOLPP) with k objective functions $z_i(x)$, i = 1. . . k as

$$\begin{aligned} &Min\ \mathrm{z(x)} = (\mathrm{z}_1(x), \mathrm{z}_2(x), ..., \mathrm{z}_k(x)) \\ &subject\ \text{to}\ \mathrm{x} \in \mathrm{X} \end{aligned} \tag{1}$$

Here $X = \{x \in \mathbb{R}^n : Ax \le b, x \ge 0\}, b = (b_1, b_2, ..., b_m) \in \mathbb{R}^m\}$ and A is a $m \times n$ matrix. By assuming that the decision maker (DM) has imprecise aspiration levels for each objective function $z_i(x)$, i = 1. . . k of model (1), several methods have been proposed in literature for characterizing Pareto optimal solutions to MOLPP (1).

One such approach involves usage of fuzzy set theory [13, 14]. Here a MOLPP is converted into single objective optimization problem to attain fuzzy efficient solution [11, 12] and then Pareto optimal solution by using the fuzziness of the DM's aspiration with

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respect to the goals of imprecise objective functions (and constraints till they are symmetric!) [17]. Recently the concepts of evolutionary techniques such as particle swarm optimization and genetic algorithm are proposed to solve multi objective optimization problems. Suitable definition of continuity and differentiability of fuzzy-valued function have also been used to study multi-objective programming problems with fuzzy-valued objective functions. More solution concepts and techniques such as the KKT optimality conditions, the necessary and sufficient conditions of Pareto-optimality for imprecise multi-objective programming problems have also been discussed and tackled with.

One such method to solve fuzzy MOLPP is two phase method [2,3]. Although the max-min operator is usually applied due to its easy computation, it does not guarantee to yield a fuzzy-efficient solution if more than one optimal solution exists [11]. The concept of two-phase procedure can also be employed to solve MOLPP with all fuzzy coefficients [10]. Sakawa contained rather comprehensive survey regarding interactive methods to MOLPP and fuzzy MOLPP [7, 8].

Recently Jimenez and Bilbao (2009) [6] showed that a fuzzy-efficient solution [11, 12] may not be a Pareto-optimal solution in the case that one of fuzzy goals is fully achieved. And they proposed a modified procedure which extended the two-phase approach of Guu and Wu (1997, 1999) [3, 4] and approach of Dubois and Fortemps (1999) [2] to attain a Pareto-optimal solution for model (1).

But according to Yan-Kuen Wu et al (2015) [15] the proposed approach by Jimenez and Bilbao (2009) [6] cannot guarantee to be a general procedure to attain Pareto-optimal solution. And accordingly another two-phase approach is proposed to directly obtain a Pareto-optimal solution. According to them, the computation of this new two-phase approach is more simplified than Jimenez and Bilbao's modified procedure. In fact, in proposed approach by Wu et al, membership function of fuzzy set theory has been redefined [15].

The paper is organized as follows. In section 2, we identify that this redefined membership function has not been used according to its definition by Wu et al (2015) [15]. We also identify that a problem may become *unnecessarily* infeasible by keeping lower bound of membership function intact. In subsection 2.1., we discuss two examples and highlight that it is not always realistic to quantify or measure each type of imprecise information with in some bounded subset of real line. In subsection 2.2., a new function, call it T-characteristic function, and then T-efficient solution is introduced. In section 3, few lemmas are discussed related to Pareto optimal solution of an MOLPP under imprecise environment. Then in section 4, we propose an algorithm to find T-efficient and finally Pareto optimal solution to a MOLPP under imprecise environment. To further explain our approach, we solve numerical examples. The second example is same as suggested by Jimenez and Bilbao in 2009 [6]. The third one is same as suggested by Zimmermann in 1978 [13]. Conclusions are drawn at last.

2. Overview on membership function of fuzzy set theory

In proposed approach by Wu et al in 2015 [15], membership function of fuzzy set theory is redefined. But their proposed mathematical models as well as numerical examples were not according to their definition of redefined membership function $\mu_i(z_i(x))$. From mathematical model as well as from those two numerical examples, it is clear that only one part of redefined membership function $\mu_i(z_i(x))$, *where objective value does not exceed sum of goal*

and tolerance (for minimization type of objective functions), was used. But these extra constraints were not added to set of existing constraints. Thus in problem formulation it happened that redefined membership function $\mu_i(z_i(x))$ was strictly monotone over entire set of real numbers where as it was not so in definition. Later example 1 is given in this paper to show that those additional constraints may make a problem infeasible.

Actually upper bound of membership function was removed by Wu et al (2015) but there still remained a lower bound at 0 [15]. In those examples by Wu et al, those additional constraints in definition of redefined membership function that *objective value does not exceed sum of goal and tolerance (for minimization type of objective functions*) were of little use [15]. But in some cases, if the lower bound of membership function remains intact, the problem may become infeasible. A solution is always better than no solution. "Half a loaf is better than none." A point to be noted.

Thus the attention must be put on membership function, now.

2.1. Membership function in fuzzy set

Historically fuzzy set theory was developed from crisp set theory to quantify imprecise information. In crisp set, the belongingness of an element is represented by characteristic function $\chi : S \to \{0,1\}$. An element belongs to a crisp subset A of universal set S if its value is 1 and does not belong to the subset A if its value is zero. On the other hand, fuzzy subset $\tilde{A}$ of universal set S can be characterized as a set of ordered pair of element x and its grade $\mu_{\tilde{A}}(x)$ and is written as $\tilde{A} = \{< x, \mu_{\tilde{A}}(x) >: x \in S\}$. Here the membership function $\mu_{\tilde{A}}$ is defined as $\mu_{\tilde{A}} : S \to [0,1]$, where S is universal set. According to Zadeh, the membership value $\mu_{\tilde{A}}(x)$ indicates the degree of belongingness of the element x to the fuzzy set $\tilde{A}$. The membership values between 0 and 1 characterize members of S that belong to the fuzzy set $\tilde{A}$ *partially*.

Let us consider few examples. Consider fuzzy set $\tilde{A}$ of all countries that have chance to win next Soccer World Cup in 2018. Most of us will agree that the chance of Russia is nil. So it can be safely stated that Russia is in $\tilde{A}$ with membership value 0. Again chance of India's winning is also nil. So India is also in $\tilde{A}$ with membership value 0. It means that Russia and India are at same level! But it is certain that even if Russia wins the 2018 Soccer World Cup, unfortunately India will certainly not be the winner. So in this case membership function of fuzzy set theory fails to explain!

Again, suppose $\tilde{A}$ is a fuzzy set representing good students. A student getting more than 75% in examination may be considered a good student. So one scoring 76% has membership value 1 in fuzzy set $\tilde{A}$ and another student scoring 98% will also have membership value 1 in fuzzy set $\tilde{A}$. Certainty, they are not same!

Further analysis highlights that the cause lies in the definition of membership function itself. Membership function in fuzzy set theory is useful and provides satisficing result when extreme ends of imprecise information can be quantified within the boundary of zero and one. But it is not always correct or feasible to quantify or measure each type of imprecise information with in some bounded subset of real line.

2.2. T-characteristic function, Pareto-optimal solution and T-efficient solution

It is noted that an element of a fuzzy set can lie *partially* or *never* lie or *must* lie in that set. Idea of lying partially is well quantified or measured by membership function of fuzzy set theory. But it does not suit in case of must lying (membership value always one) and never lying (membership value always zero).

Again in the case of decision making under fuzzy environment [1], one interesting and useful property of membership function is that higher value of membership function always gives better result of objective function. And there is not only membership function but also many such functions with this amazing characteristic.

Therefore one new function, say T-characteristic function, that handles these underlying issues well, is defined.

Definition 1. Let S denotes the universal set. Then T-characteristic function is denoted by $T_{\tilde{A}}(x)$ and is defined as $T_{\tilde{A}} : S \to \mathbb{R}$ that assigns a real number $T_{\tilde{A}}(x)$ to each element $x \in S$, where $T_{\tilde{A}}(x)$ represents the grade of membership of $x \in S$ in $\tilde{A}$.

As for example, in decision making under imprecise environment [1], in case of maximization type of objective function z_1; one may simply consider the T-characteristic function as z_1 itself. And in case of minimization type of objective function z_2; one may consider T-characteristic function as (–) z_2. They satisfy the same property and can be used to convert a MOLPP into single objective optimization problem to obtain Pareto optimal solution. Next in imprecise environment, Pareto optimal solution and T-efficient solution are defined as follows:

Definition 2. A decision plan $x_0 \in X$ is said to be a Pareto-optimal solution to the MOLP model (1) if there does not exist another $y \in X$ such that $z_i(y)) \le z_i(x_0)$ for all i, i≠j and $z_j(y) < z_j(x_0)$ for at least one j.

Definition 3. A decision plan $x_0 \in X$ is said to be a T-efficient solution to the MOLP model (1) if there does not exist another $y \in X$ such that $T_i(z_i(x_0)) \le T_i(z_i(y))$ for all i, i≠j and $T_j(z_j(x_0)) < T_j(z_j(y))$ for at least one j.

3. Obtaining T-efficient and Pareto-optimal solution

Within the scope of the multi-objective decision making (MODM) theory, the Pareto-optimality of solution is a necessary condition in order to guarantee the rationality of a decision. Then using the method as discussed by Bellman and Zadeh, model (1) is equivalent to

$$\begin{aligned} &\max\ (T_1(z_1(x)), T_2(z_2(x)), \ldots, T_k(z_k(x))) \\ &s.t.\ \mathrm{x} \in \mathrm{X} \end{aligned} \tag{1a}$$

And model (1a) is equivalent to

$$Min\ (T_1(z_1(x)), T_2(z_2(x)), ..., T_k(z_k(x)))^{-1}$$
$$s.t.\ x \in X \qquad (1b)$$

Using the min-max method by Bowman (1976) [16], this MOLPP (1b) may be rewritten as

$$Min \max_i (T_i(z_i(x)))^{-1}, i = 1, 2...k$$
$$s.t.\ x \in X \qquad (1c)$$

Let $v = \max_i (T_i(z_i(x)))^{-1}, i = 1, 2...k$. Then model (1c) reduces to

$$\min\ v$$
$$\text{s.t.}$$
$$(T_i(z_i(x)))^{-1} \le v, i = 1, 2...k$$
$$x \in X,\ \text{v unrestricted in sign} \qquad (1d)$$

If we put $v = 1/\lambda$, then (1d) reduces to

$$\max\ \lambda$$
$$\text{s.t. } T_i(z_i(x)) \ge \lambda, i = 1, 2, ..., k$$
$$x \in X,\ \lambda \text{ unrestricted in sign} \qquad (2)$$

The relationships between the optimal solution x* of the problem (2) and the Pareto optimal solution to the MOLPP (1) can be characterized by following lemmas.

Lemma 1. *If x* ∈ X is unique optimal solution to the max-min problem (2), then x* is a Pareto optimal solution to the MOLPP (1).*

Proof. Let x* be unique optimal solution to problem (2). If possible let x* is not a Pareto optimal solution to problem (1). Then there exist at least one $y \in X$ such that $z_i(y)) \le z_i(x^*)$ for all i, i ≠ j and $z_j(y) < z_j(x^*)$ for at least one j. Since each $T_i(z_i)$ is strictly monotonic decreasing function for minimizing type of objective functions z_i, i= 1, 2…k, we get

$$T_i(z_i(x^*)) \le T_i(z_i(y)) \text{ for all i =1, 2,...,k , i } \ne j$$
$$and\ T_j(z_j(x^*)) < T_j(z_j(y)).$$

It contradicts that x* is *unique* optimal solution to problem (2). Therefore our assumption is wrong. Hence x* is Pareto optimal solution to MOLPP■

Lemma 2. *If x* ∈ X is a Pareto optimal solution of the MOLPP, then x* is an optimal solution of the problem (2) for some fi, i=1, 2…k.*

Proof. Let x* is Pareto optimal solution to MOLPP. If possible let x* is not an optimum solution to the minimax problem for any $T_{\tilde{A}^T i}$, i = 1, 2…k.

Then for T_i, $i = 1,\ldots k$, there exists $x \in X$ such that $T_j(z_j(x)) > T_j(z_j(x^*)) = \lambda(x^*)$, for some j and $T_i(z_i(x)) \geq T_i(z_i(x^*)) = \lambda^*(x^*), i = 1,...k, i \neq j.$

But T_i is strictly decreasing function for each i = 1…k. Hence it gives $z_j(x) < z_j(x^*)$ for some j and $z_i(x) \leq z_i(x^*), i = 1,2,...k, i \neq j.$ It contradicts that x* is Pareto optimal solution to the MOLPP. Therefore our assumption is wrong. Hence x* is an optimal solution to the minimax problem (2) for T_i, i =1, 2… k■

From lemma 1 and lemma 2, if the uniqueness of the optimal solution x* for the minimax problem (2) is not guaranteed, it is necessary to perform the Pareto optimality test of x*.

The Pareto optimality test for x* can be performed by solving the following linear programming problem with the decision variables $x = (x_1, x_2, ...x_n)^T$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, ...\varepsilon_k)^T$ [2,3].

$$\max \sum_{i=1}^{k} \varepsilon_i$$

$$T_i(z_i(x)) - \varepsilon_i \geq T_i(z_i(x^*)), i = 1,2,...,k$$

$$x \in X, \varepsilon_i \geq 0, i = 1,2,...,k. \tag{3}$$

Lemma 3. Let $\bar{x}$ and $\bar{\varepsilon}$ be optimal solution to problem (3). Then

I. If all $\bar{\varepsilon}_i = 0$, then x* is Pareto optimal solution of the MOLPP.

II. If at least one $\bar{\varepsilon}_i > 0$, then x* is not Pareto optimal solution of the MOLPP. Instead of x*, $\bar{x}$ is Pareto optimal solution to the MOLPP.

Proof of this lemma is already recorded in literature [3, 4]. It may be noted that several authors have proposed other procedures to test Pareto optimality and can be used in place of model (3).

4. A general framework for solving multi-objective programming problems under imprecise environment

The above ideas can be integrated into a general framework or algorithm to obtain a Pareto-optimal solution to the MOLPP which has the additional property of being T-efficient for chosen T-characteristic functions. The steps of the proposed algorithm can be synthesized as follows [9, 10]

Step 1: Define suitable T-characteristic function for each imprecise objective function in such a way that higher value of T-characteristic function always gives better value of objective function. Well-defined T-characteristic functions preserve this characteristic and gives finite value always.

Step 2: Construct the max-min problem as model (2).

Step 3: Solve the max-min problem and obtain the T-efficient solution x* of it.

Step 4: To test whether this T-efficient solution x* is Pareto optimal, solve model (3).

Step 5: Let $\bar{x}$ and $\bar{\varepsilon}$ be optimal solution of model (3) in step 4. Then two cases may arise:

a) If $\overline{\varepsilon}_i = 0 \ \forall$ i, then x* is Pareto optimal solution of the MOLPP.

b) If $\overline{\varepsilon}_j > 0$ for at least one j, then x* is not Pareto optimal solution of the MOLPP. Instead of x*, $\overline{x}$ is Pareto optimal solution to the MOLPP.

Step 6: This solution is T-efficient as well as Pareto-optimal. The algorithm is complete.

4.1. Numerical examples

To check the usefulness of proposed approach, consider these numerical examples. The first example is chosen to show that the addition of constraints, necessary to make redefined membership function consistent with its definition, as proposed by Wu et al [15], may make problem infeasible. The second one is the same as suggested by Jimenez and Bilbao in 2009 [6]. The third one is the same as suggested by Zimmermann in 1978 [13].

4.1.1. Example 1

Let us consider a fuzzy multi objective linear programming problem as

$$\begin{aligned}
&\text{fuzzy } max \quad 5x_1 + 5x_2 \\
&\text{fuzzy } min \quad 5x_1 + x_2 \\
&\text{fuzzy max} \quad 3x_1 - 8.2x_2 \\
&s.t. \\
&5x_1 + 7x_2 \le 12, \quad 9x_1 + x_2 \le 10, \quad -5x_1 + x_2 \le 3, \\
&x_1, x_2 \ge 0
\end{aligned}$$

4.1.1.1. Using redefined membership function

Suppose that the DM specifies goal and tolerance for each of the imprecise objective functions as follows:

Objective function	Goal	Goal plus Tolerance
Fuzzy Maximum	10	9
Fuzzy Minimum	0	1
Fuzzy Maximum	3	-7

Table 1. *Goal and tolerance of fuzzy objective functions*

It must be noted that these values of goal and goal plus tolerance of each of the objective functions are within their individual maximum and minimum values. Based on this information, as Wu et al [15], redefined membership functions are defined as

$$\mu_1(z_1(x)) = \begin{cases} \dfrac{5x_1 + 5x_2 - 9}{1}, & 5x_1 + 5x_2 \ge 9 \\ 0 & ,5x_1 + 5x_2 \le 9 \end{cases}$$

$$\mu_2(z_2(x)) = \begin{cases} \dfrac{1-(5x_1+x_2)}{1}, & 5x_1+x_2 \le 1 \\ 0, & 5x_1+x_2 \ge 1 \end{cases}$$

$$\mu_3(z_3(x)) = \begin{cases} \dfrac{3x_1-8.2x_2-3}{10}, & 3x_1-8.2x_2 \ge -7 \\ 0, & 3x_1-8.2x_2 \le -7 \end{cases}$$

The max-min problem, using the redefined membership function *properly* (according to their definition), is as follows:

$$\begin{aligned}
&\max\ \lambda \\
&\text{s.t.} \\
&\frac{5x_1+5x_2-9}{1} \ge \lambda, \quad \frac{1-(5x_1+x_2)}{1} \ge \lambda, \quad \frac{3x_1-8.2x_2-3}{10} \ge \lambda, \\
&5x_1+5x_2 \ge 9, \quad 5x_1+x_2 \le 1, \quad 3x_1-8.2x_2 \ge -7, \\
&5x_1+7x_2 \le 12, \quad 9x_1+x_2 \le 10, \quad -5x_1+3x_2 \le 3, \\
&x_1, x_2 \ge 0, \lambda \text{ is unrestricted in sign}
\end{aligned}$$

Using Lingo 15.0.32 to solve this problem, we find that the problem has no feasible solution.

4.1.1.2. Using T-characteristic function

Now the T-characteristic functions for each of the objective are formed, keeping in mind that the higher value of T-characteristic function gives better value of corresponding objective function in each case as

$$T_1(z_1(x)) = \frac{5x_1+5x_2-9}{1}, x_1, x_2 \ge 0,$$

$$T_2(z_2(x)) = \frac{1-(5x_1+x_2)}{1}, x_1, x_2 \ge 0,$$

$$T_3(z_3(x)) = \frac{3x_1-8.2x_2-3}{10}, x_1, x_2 \ge 0$$

If these T-characteristic functions are used instead of membership functions, the problem in model (2) may be formulated as

$$\begin{aligned}
&\max\ \lambda \\
&\text{s.t.} \\
&\frac{5x_1+5x_2-9}{1} \ge \lambda, \quad \frac{1-(5x_1+x_2)}{1} \ge \lambda, \quad \frac{3x_1-8.2x_2-3}{10} \ge \lambda, \\
&5x_1+7x_2 \le 12, \quad 9x_1+x_2 \le 10, \quad -5x_1+3x_2 \le 3, \\
&x_1, x_2 \ge 0, \lambda \text{ is unrestricted in sign}
\end{aligned}$$

Using Lingo 15.0.32 to solve this problem, the optimal solution is obtained as

$$\lambda^* = -1.333333, x_1^* = 0.2, x_2^* = 1.333333, z_1^*(x^*) = 7.665, z_2^*(x^*) = 2.333, z_3^*(x^*) = -10.331,$$
$$T_1^*(z_1^*(x^*)) = -1.333333, T_2^*(z_2^*(x^*)) = -1.333333, T_3^*(z_3^*(x^*)) = -1.333333.$$

It is observed that values of T-characteristic functions are negative. To test the Pareto optimality of this solution, construct model (3) for this problem as follows

$$\max \sum_{i=1}^{3} \varepsilon_i$$
subject to
$$\frac{5x_1 + 5x_2 - 9}{1} - \varepsilon_1 \geq -1.333333,$$
$$\frac{1-(5x_1 + x_2)}{1} - \varepsilon_2 \geq -1.333333,$$
$$\frac{3x_1 - 8.2x_2 - 3}{10} - \varepsilon_3 \geq -1.333333,$$
$$5x_1 + 7x_2 \leq 12, \quad 9x_1 + x_2 \leq 10, \quad -5x_1 + 3x_2 \leq 3,$$
$$x_i, \varepsilon_i \geq 0, \; i = 1, 2, 3.$$

Let $\overline{x}$ and $\overline{\varepsilon}$ be optimal solution to this problem. Using Lingo 15.0.32 to solve this problem, the optimal solution is obtained as follows:

$$\overline{\varepsilon}_1 = \overline{\varepsilon}_2 = \overline{\varepsilon}_3 = 0, \overline{x}_1 = 0.2, \overline{x}_2 = 1.333333, \overline{z}_1(\overline{x}) = 7.665, \overline{z}_2(\overline{x}) = 2.333, \overline{z}_3(\overline{x}) = -10.331.$$

Here all $\overline{\varepsilon}_i = 0.$ Hence the solution x* obtained in previous stage is itself Pareto optimal solution. It is observed that the *proper* usage of definition of redefined membership function, as proposed by Wu et al (2015) [15], makes this problem infeasible, whereas Pareto optimal solution is found by using proposed T-characteristic function.

4.1.2. Example 2

To illustrate the proposed method further, consider the numerical example suggested by Jimenez and Bilbao (2009) [6] as follows:

$$\text{Minimize } z_1(x) = 3x_1 + 3x_2 + 3x_3$$
$$\text{Minimize } z_2(x) = 2x_1 + x_2 + 2x_3$$
$$\text{Minimize } z_3(x) = 4x_1 + 4x_2 + 2x_3$$
$$\text{subject to } 4x_1 + 2x_2 + 4x_3 \geq 18$$
$$x_1 \geq 1, \quad x_2 \geq 0, \quad 0 \leq x_3 \leq 3$$

4.1.2.1. T-characteristic function similar to membership function of fuzzy set theory

The proposed approach to solve this MOLPP with fuzzy goals $g_1 = 21, g_2 = 8, g_1 = 13$ and tolerances $t_1 = 3, t_2 = 2, t_3 = 2$ is employed. Based on proposed definition, the T-characteristic functions (partly similar to membership function) may be defined as follows:

$$T_1(z_1(x)) = \frac{24-(3x_1+3x_2+3x_3)}{3}, x_1, x_2, x_3 \geq 0 \quad T_2(z_2(x)) = \frac{10-(2x_1+x_2+2x_3)}{2}, x_1, x_2, x_3 \geq 0$$
$$T_3(z_3(x)) = \frac{15-(4x_1+4x_2+2x_3)}{2}, x_1, x_2, x_3 \geq 0$$

Then the model (2) can be formulated as below:

$$\max \ \lambda$$
subject to the constraints
$$\frac{24-(3x_1+3x_2+3x_3)}{3} \geq \lambda, \quad \frac{10-(2x_1+x_2+2x_3)}{2} \geq \lambda, \quad \frac{15-(4x_1+4x_2+2x_3)}{2} \geq \lambda,$$
$$4x_1+2x_2+4x_3 \geq 18, \quad x_1 \geq 1, \quad 0 \leq x_3 \leq 3,$$
$x_2 \geq 0, \lambda$ is unrestricted in sign

Using Lingo 15.0.32 to solve this problem, the optimal solution is obtained as

$$\lambda^* = 0.5, x_1^* = 2.5, x_2^* = 0, x_3^* = 2, z_1^*(x^*) = 13.5, z_2^*(x^*) = 9, z_3^*(x^*) = 14,$$
$$T_1^*(z_1^*(x^*)) = 3.5, T_2^*(z_2^*(x^*)) = 0.5, T_3^*(z_3^*(x^*)) = 0.5.$$

To test the Pareto optimality of this solution, construct model (3) of this problem as follows

$$\max \sum_{i=1}^{3} \varepsilon_i$$
subject to the constraints
$$\frac{24-(3x_1+3x_2+3x_3)}{3} - \varepsilon_1 \geq 3.5,$$
$$\frac{10-(2x_1+x_2+2x_3)}{2} - \varepsilon_2 \geq 0.5$$
$$\frac{15-(4x_1+4x_2+2x_3)}{2} - \varepsilon_3 \geq 0.5$$
$$4x_1+2x_2+4x_3 \geq 18, \quad x_1 \geq 1, \quad 0 \leq x_3 \leq 3,$$
$$x_2, \varepsilon_i \geq 0, i = 1,2,3$$

Let $\bar{x}$ and $\bar{\varepsilon}$ be optimal solution to this problem. Using Lingo 15.0.32 to solve this problem, we obtain the optimal solution as

$$\bar{\varepsilon}_1 = \bar{\varepsilon}_2 = 0, \bar{\varepsilon}_3 = 1, \bar{x}_1 = 1.5, \bar{x}_2 = 0, \bar{x}_3 = 3, \bar{z}_1(\bar{x}_1) = 13.5, \bar{z}_2(\bar{x}_2) = 9, \bar{z}_3(\bar{x}_3) = 12.$$

It is observed that $\bar{z}_3 \prec z_3^*$. And according to lemma 3, $\bar{x}$ is the Pareto optimal solution. It is further noted that this Pareto optimal solution $\bar{x}$ is same as solution obtained by Wu et al [15].

4.1.2.1.1. Another T-characteristic function

Again the T-characteristic functions may be defined for minimization type of objective function as

$$T_1(z_1(x)) = -(3x_1 + 3x_2 + 3x_3), x_1, x_2, x_3 \geq 0$$

$$T_2(z_2(x)) = -(2x_1 + x_2 + 2x_3), x_1, x_2, x_3 \geq 0$$

$$T_3(z_3(x)) = -(4x_1 + 4x_2 + 2x_3), x_1, x_2, x_3 \geq 0$$

Based on these, model (2) can be formulated as

$$\begin{aligned}
&\max\ \lambda \\
&\text{subject to the constraints} \\
&-(3x_1 + 3x_2 + 3x_3) \geq \lambda, \quad -(2x_1 + x_2 + 2x_3) \geq \lambda, \quad -(4x_1 + 4x_2 + 2x_3) \geq \lambda \\
&4\mathrm{x}_1 + 2\mathrm{x}_2 + 4\mathrm{x}_3 \geq 18, \quad \mathrm{x}_1 \geq 1, \quad 0 \leq \mathrm{x}_3 \leq 3, \\
&x_2 \geq 0, \lambda \text{ is unrestricted in sign}
\end{aligned}$$

Using Lingo 15.0.32 to solve this problem, the optimal solution is obtained as

$$\begin{aligned}
&\lambda^* = \text{-}13.5, x_1^* = 1.5, x_2^* = 0, x_3^* = 3, z_1^* = 13.5, z_2^* = 9, z_3^* = 12, \\
&T_1^*(z_1^*(x^*)) = -13.5, T_2^*(z_2^*(x^*)) = -9, T_3^*(z_3^*(x^*)) = -12.
\end{aligned}$$

To test the Pareto optimality of this problem, construct model (3) of this problem as follows

$$\begin{aligned}
&\max \sum_{i=1}^{3} \varepsilon_i \\
&\text{subject to the constraints} \\
&-(3x_1 + 3x_2 + 3x_3) - \varepsilon_1 \geq -13.5 \\
&-(2x_1 + x_2 + 2x_3) - \varepsilon_2 \geq -9 \\
&-(4x_1 + 4x_2 + 2x_3) - \varepsilon_3 \geq -12 \\
&4\mathrm{x}_1 + 2\mathrm{x}_2 + 4\mathrm{x}_3 \geq 18, \quad \mathrm{x}_1 \geq 1, \quad 0 \leq \mathrm{x}_3 \leq 3, \\
&x_2, \varepsilon_i \geq 0, i = 1, 2, 3.
\end{aligned}$$

Let $\overline{x}$ and $\overline{\varepsilon}$ be optimal solution to this problem. Using Lingo 15.0.32 to solve this problem, obtain the optimal solution as

$$\begin{aligned}
&\overline{\varepsilon}_1 = \overline{\varepsilon}_2 = \overline{\varepsilon}_3 = 0, \overline{x}_1 = 1.5, \overline{x}_2 = 0, \overline{x}_3 = 3, \overline{z}_1 = 13.5, \overline{z}_2 = 9, \overline{z}_3 = 12, \\
&\overline{T}_1(\overline{z}_1(\overline{x})) = -13.5, \overline{T}_2(\overline{z}_2(\overline{x})) = -9, \overline{T}_3(\overline{z}_3(\overline{x})) = -12.
\end{aligned}$$

Here all $\overline{\varepsilon}_i = 0.$ Hence the solution obtained in previous stage is itself Pareto optimal solution.

4.1.3. Example 3

A company manufactures two products 1 and 2 on given capacities. Product 1 yields of profit of 2 $ per piece and product 2 of 1 $ per piece. While product 2 can be exported, yielding a revenue of 2 $ per piece in foreign countries, product 1 needs imported raw materials of 1 $ per piece. Two goals are established: (a) Profit maximization and (b)

Maximum improvement of the balance of trade, i.e. maximum difference of exports minus imports.

Then the problem can be modelled as

$$\begin{aligned}
&\max imize\ \mathrm{z}_1(\mathrm{x}) = -\mathrm{x}_1 + 2\mathrm{x}_2 \\
&\text{maximize } \mathrm{z}_2(\mathrm{x}) = 2\mathrm{x}_1 + \mathrm{x}_2; \\
&subject \text{ to} \\
&-\mathrm{x}_1 + 3\mathrm{x}_2 \le 21, \quad \mathrm{x}_1 + 3\mathrm{x}_2 \le 27, \quad 4\mathrm{x}_1 + 3\mathrm{x}_2 \le 45 \\
&3\mathrm{x}_1 + \mathrm{x}_2 \le 30, \quad \mathrm{x}_1, \mathrm{x}_2 \ge 0
\end{aligned}$$

Suppose we construct T-characteristic function $T(z(x))$ of objective function z(x) of maximization type (partly similar to membership function of fuzzy set theory, as used by Zimmermann) as

$$T_1(z_1(x)) = \frac{z_1+3}{17}, x_1, x_2 \ge 0, \quad T_2(z_2(x)) = \frac{z_2-7}{14}, x_1, x_2 \ge 0.$$

Based on these functions, model (2) can be formulated as follows:

$$\begin{aligned}
&\max\ \lambda \\
&\text{subject to} \\
&\frac{-\mathrm{x}_1 + 2\mathrm{x}_2 + 3}{17} \ge \lambda, \quad \frac{2\mathrm{x}_1 + \mathrm{x}_2 - 7}{14} \ge \lambda, \\
&-\mathrm{x}_1 + 3\mathrm{x}_2 \le 21, \quad \mathrm{x}_1 + 3\mathrm{x}_2 \le 27, \\
&4\mathrm{x}_1 + 3\mathrm{x}_2 \le 45, \quad 3\mathrm{x}_1 + \mathrm{x}_2 \le 30, \\
&\mathrm{x}_1, \mathrm{x}_2 \ge 0, \lambda \text{ is unrestricted in sign}
\end{aligned}$$

Using Lingo 15.0.32 to solve this problem, the optimal solution is obtained as

$$\begin{aligned}
&\lambda^* = 0.74, x_1^* = 5.03, x_2^* = 7.32, z_1^* = 9.61, z_2^* = 17.39, \\
&T_1^*(z_1^*(x^*)) = 0.74, T_2^*(z_2^*(x^*)) = 0.74.
\end{aligned}$$

To test the Pareto optimality of this problem, construct model (3) of this problem as follows

$$\begin{aligned}
&\max \sum_{i=1}^{2} \varepsilon_i \\
&\text{subject to} \\
&\frac{-\mathrm{x}_1 + 2\mathrm{x}_2 + 3}{17} \ge \lambda, \ \frac{2\mathrm{x}_1 + \mathrm{x}_2 - 7}{14} \ge \lambda, \\
&-\mathrm{x}_1 + 3\mathrm{x}_2 \le 21, \ \mathrm{x}_1 + 3\mathrm{x}_2 \le 27, \\
&4\mathrm{x}_1 + 3\mathrm{x}_2 \le 45, \ 3\mathrm{x}_1 + \mathrm{x}_2 \le 30, \\
&\mathrm{x}_i, \varepsilon_i \ge 0, i = 1, 2.
\end{aligned}$$

Let $\overline{x}$ and $\overline{\varepsilon}$ be optimal solution to this problem. Using Lingo 15.0.32 to solve this problem, obtain the optimal solution as

$$\overline{\varepsilon}_1 = \overline{\varepsilon}_2 = 0, \overline{x}_1 = 5.03, \overline{x}_2 = 7.32, \overline{z}_1 = 9.61, \overline{z}_2 = 17.39,$$
$$\overline{T}_1(\overline{z}_1(\overline{x})) = 0.74, \overline{T}_2(\overline{z}_2(\overline{x})) = 0.74.$$

Here all $\overline{\varepsilon}_i = 0.$ Hence the solution obtained in previous stage is itself Pareto optimal solution. The maximum degree of "overall satisfaction" (0.74) is achieved for the solution $\overline{x}$ = $(5.03, 7.32)^{T}$. And interestingly this result is exactly matching with result obtained by Zimmermann in 1978 [13].

5. Conclusions

In this paper, we identify that the redefined membership function, proposed by Wu et al (2015) [15], was not used according to definition in their paper. It is *not* strictly monotonic over its entire domain in the definition (lower bound zero) but is so in problem formulation as well as in those two numerical examples (constraint showing lower bound is not present).

In this paper we further identify that although this redefined membership function correctly identifies *the best solution among the good; it fails to identify bad solution among the worst!* As shown in ex. 1 of our paper, proper usage of redefined membership function may make the MOLPP infeasible.

Moreover in real life situations, DM may not be able to identify the goal or tolerance or both; and in some cases, a DM may not exist at all. In these scenarios, it is problematic or even impossible to construct fuzzy membership function or that redefined membership function.

Further we note that the primary objective is to identify a function such that higher value of function will generate better value for objective. Hence in decision making under uncertainty, we propose to define and use T-characteristic function covering the entire set of possible solutions in place of membership function. And this is a general procedure to obtain a Pareto-optimal solution that has the additional property of being T-efficient.

And in fact the usage of a well-defined T-characteristic function also removes the need to collect information from DM and saves precious time of both.

From this we also conclude that, under these circumstances, the issue of getting no solution even by using fuzzy set theory yielded from several published methods seems worthwhile or even necessary to reconsider. The same is also true for nonlinear problems of this type.

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